Newtonian approach for the Kepler-Coulomb problem from the point of view of velocity space

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Abstract

The hodograph of the Kepler-Coulomb problem, that is, the path traced by its velocity vector, is shown to be a circle and then it is used to investigate the properties of the motion. We obtain the configuration space orbits of the problem starting from initial conditions given using nothing more than the methods of synthetic geometry so close to Newton's approach. The method works with elliptic, parabolic and hyperbolic orbits; it can even be used to derive Rutherford's relation from which the scattering cross section can be easily evaluated. We think our discussion is both interesting and useful inasmuch as it serves to relate the initial conditions with the corresponding trajectories in a purely geometrical way uncovering in the process some interesting connections seldom discussed in standard treatments.

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Resumen

Demostramos que la hodógrafa del problema de Kepler-Coulomb, esto es, la trayectoria que sigue su vector velocidad, es una circunferencia y la usamos para establecer geométricamente otras propiedades del movimiento. Obtenemos la relación entre la hodógrafa y la órbita en el espacio de las configuraciones para el problema de Kepler-Coulomb partiendo de condiciones iniciales dadas y empleando nada más que los métodos de la geometría sintética tan caros a Newton. El método que proponemos incluye tanto a las órbitas elípticas, como a las parabólicas e hiperbólicas y puede también usarse para deducir la relación de Rutherford, la que es la clave para obtener la sección eficaz de dispersión. Pensamos que nuestro enfoque es tanto interesante como útil ya que permite trazar las trayectorias usando sólo métodos geométricos y, además, reconocer algunas relaciones que no son evidentes en los tratamientos más usuales.

Classification Numbers: 03.20.+i, 95.10.C

1. Introduction

We have been analysing an approach to solve the Kepler-Coulomb problem employing the properties of its hodograph and its relationship to their orbits [1–4]. Please recall that *hodograph* is the name given to the path traced by the velocity vector of a system in velocity space. In this work we purport to express all our arguments in geometric terms in a sort of Newtonian fashion. The hodograph as a mean towards understanding the dynamics of a system was introduced by Hamilton [5] —he even invented the term— during the last century. Hamilton was able to show that the hodograph under an inverse squared centre of force [5–7] or, as we call it here, in the Kepler-Coulomb problem is always a circle. It is curious to notice that Hamilton proved that, in a way, the ancient Greek astronomers were right, the motion of planets around the sun is indeed circular, they just got wrong the space since the hodograph inhabits velocity rather than configuration space. However, even if one knows that the Kepler-Coulomb hodograph is circular in shape, it is natural to wonder how can that circle be related with the well-known orbits in configuration space. Let us note that the problem is easily solved in an analytical treatment since we can use the polar angular coordinate in the plane of the orbit, θ, for relating the trajectory in v-space with the trajectory in r-space [2–4]. Furthermore, let us point out that the problem posed has been already solved geometrically, since there exist beautiful methods developed by Maxwell [6] and by Feynman [8] to solve it.

In this work we discuss a —we expect— novel geometric approach to the relationship between the hodograph and the orbit of the Kepler-Coulomb problem. We begin establishing the circular shape of the problem's hodograph using standard analytical methods and then rework the path from the hodograph to the orbit using techniques that —we think— are akin to those in the Principia [9]. In our view, this geometric approach uncovers the geometrical beauty associated with the physics of the problem which no doubt contributed to the attraction felt towards it by many people trough the centuries, from the ancient Mayan astronomers to their modern counterparts. We do think this type of approach contributes to a better understanding of the interplay between the geometry and the physical properties of the solution to the Kepler-Coulomb problem. It is to be noted that, mostly, the constructions presented here require no more than straight edge and compass to be realized. But, before embarking in the discussion, let us convene that the trajectory in configuration space be always called the *orbit* whereas the trajectory in velocity space be always called the *hodograph*.

2. The hodograph of the Kepler-Coulomb problem

The equation of motion of a particle interacting with an inverse squared centre of force is,

$$m\frac{d\mathbf{v}}{dt} = -\frac{\alpha}{r^2}\hat{\mathbf{e}}_r, \tag{1}$$

where m, α, $\mathbf{v}$ and $\hat{\mathbf{e}}_r$ are, respectively, the mass of the particle, a constant characterizing the interaction strenght (which is positive if the interaction is attractive or negative if it is repulsive), the velocity vector, and the unit vector in the radial direction in configuration space. In the Kepler-Coulomb problem, described by equation (1), the energy E and the angular momentum $\mathbf{L} = m\mathbf{r} \times \mathbf{v} = mr^2\dot{\theta}\hat{\mathbf{e}}_z = L\hat{\mathbf{e}}_z$ are conserved (we choose the direction of $\mathbf{L}$ as z-axis). The motion is thus seen to be confined to a plane orthogonal to $\mathbf{L}$. In this orbital plane we may choose a polar coordinate system with unit vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r$, for describing the motion. Given this information, showing that the hodograph is a circle is not difficult, this has been done by Fano and Fano [10] using a nice but not completely geometrical approach. As posing and solving dynamical problems in geometrical terms is, however, very unfamiliar to modern readers, we have decided to start the discussion using conventional differential equation techniques; according to this, we begin by giving an standard proof of the hodograph's main properties [2]:

If we multiply (1) times L, the equation of motion becomes

$$Lm\frac{d\mathbf{v}}{dt} = -\frac{\alpha}{r^2}(mr^2\dot{\theta})\hat{\mathbf{e}}_r = \alpha m\dot{\hat{\mathbf{e}}}_\theta, \tag{2}$$

where we used $\dot{\hat{\mathbf{e}}}_\theta = -\dot{\theta}\hat{\mathbf{e}}_r$. From (2), it must be clear that the Hamilton vector [2,11]

$$\mathbf{h} = \mathbf{v} - \frac{\alpha}{L}\hat{\mathbf{e}}_\theta \tag{3}$$

is a constant of motion in the Kepler-Coulomb problem. As can be seen in this equation, the Hamilton vector is always parallel to the velocity at pericentre $\mathbf{v}_p$ [4, 12]. The magnitude of the Hamilton vector

$$h = \sqrt{\frac{2E}{m} + \frac{\alpha^2}{L^2}}, \tag{4}$$

increases when the energy E increases or when the magnitude of the angular momentum L decreases. Moreover, as follows from (3), the velocity lays in a circular arc with radius $R_h \equiv \alpha/L$ and whose centre is at the tip of $\mathbf{h}$ in velocity space. The Coulomb-Kepler *hodograph* is a circle whose centre is at the tip of $\mathbf{h}$ and, therefore, the Hamilton vector defines a dynamical symmetry axis of the hodograph —dynamical symmetry in the sense that it is not only a geometrical property, the interaction intervenes directly; for comparison note that the rest of the diameters are just geometric symmetry axes. This property of the hodograph shows that the orbit has also a dynamical symmetry axis; such axis is found by geometric means in section 3 below.

As the hodograph is a closed curve —at least when it happens to be the whole circle, i.e. in precisely the case of bounded orbits— then all the bounded orbits of the problem have to be necessarily periodic. How are other features of the hodograph

related to the properties of the orbit? As we exhibit below in sections 4.1 to 4.5, the geometric shape and the bounded or unbounded nature of the orbits change according to where the v-space origin is positioned in relation to the hodograph. Many of these features are discussed in modern language for the case of an attractive interaction in [2] and for the scattering case in [3–4].

3. From the initial conditions to the hodograph

If we are given the position $\mathbf{r}_0$ and the velocity $\mathbf{v}_0$ at a certain time t_0, how can we construct the Hamilton vector and the hodograph? In this section we show how this can be done using a very simple geometrical construction. Before beginning with the geometrical construction, we first need to calculate the lenght of the angular momentum vector: $L = mr_0 v_0 \sin\delta$, where $0 \leq \delta \leq \pi$ is the angle between the initial position and velocity. But, L is just the area of the rectangle spanned by r_0 and the component of $\mathbf{v}_0$ orthogonal to $\mathbf{r}_0$ times m, that is, it is twice the areal velocity of Kepler second law. We also need the 'length' $R_h = \alpha/L$ —remember that this ratio has really dimensions of velocity.

To understand the geometrical construction that follows it is convenient to keep figure 1 in sight. Let the point Q be the position of the centre of force. Draw the line segment $\overline{QR}$ corresponding to the initial position $\mathbf{r}_0$ (in fact, this is always the name given to the line segment representing the initial position in all the discussions that follow). Extend the segment $\overline{QR}$ up to an arbitrary point O —this just corresponds to choosing the origin in velocity space. From the v-space origin O, draw the line segment $\overline{OP}$ corresponding to $\mathbf{v}_0$ ($\overline{OP}$ is always the name of the line segment representing the initial velocity in all the discussions that follow) and draw, perpendicular to $\overline{QR}$, a line segment $\overline{OO'}$ of lenght R_h —that is, we are drawing $-\alpha/L\,\hat{\mathbf{e}}_\theta$ (recall that we defined $\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r$, where $\hat{\mathbf{e}}_z \equiv \mathbf{L}/L$). Notice that the previous construction assumes both an attractive interaction and all the conventions mentioned. To analyse a repulsive interaction the point O' had to be chosen in the opposite direction (i.e. in such case we should draw $+\alpha/L\,\hat{\mathbf{e}}_\theta$).

Using the parallelogram rule, sum $\overline{OP}$ to $\overline{OO'}$ to get the point C. The line segment $\overline{OC}$ represents the Hamilton vector. Having obtained $\mathbf{h}$, to get the hodograph draw, with centre at C, a circle of radius R_h; this represents the hodograph.

The above geometrical construction besides giving $\mathbf{h}$ and the hodograph tell us about the bounded or unbounded nature of the orbit. It is only a matter of noticing whether O is located inside the circle of the hodograph or not; if it is inside, the orbit is bounded and the energy has to be negative, if not, the orbit is unbounded and the energy is positive. Figure 1 illustrates a case in which O is inside, that is, a motion with $E < 0$. What about the case $E = 0$? As it is easy to see from (4), or just from the continuity of the descriptions, this case only happens when O sits precisely on the circle, that is, when $h = R_h = \alpha/L$ [2].

It is also easy to obtain the dynamical symmetry axis of the orbit from the

given initial conditions. We just need to draw the line segment $\overline{QS}$, which is a line perpendicular to $\overline{OC}$ passing through the centre of force Q. This follows from the paralellism of $\mathbf{h}$ and the velocity at pericentre $\mathbf{v}_p$. The line $\overline{QS}$ so drawn, *is the orbit's dynamical symmetry axis*. Notice also that $\mathbf{v}_p$ can be drawn by simply prolonguing the segment $\overline{OC}$ until it intersects the hodograph. This intercept is marked X in figure 1. If, as happens in figure 1, there are two intersections with the hodograph and not just one, the velocity space origin O is, necessarily, inside the hodograph, that is, the energy is necessarily negative. The second intercept, labeled X' in figure 1, defines the segment $\overline{OX'}$ corresponding to the velocity at the *apocentre* of the orbit, that is, to the point on the orbit farthest from the centre of force and therefore with the least magnitude. A such point obviously does not exist in the $E > 0$ case when O is outside the hodograph.

In all the section 4, we assume that the symmetry axis has been drawn as described before the geometric discussion begins. The just found dynamical symmetry axis corresponds to the direction of the Laplace-Runge-Lenz vector $\boldsymbol{\mathcal{A}} = \mathbf{h} \times \mathbf{L}$ [13–14] which always points towards the pericentre of the orbit.

4. From the hodograph to the orbit

In this section we show how given the hodograph, constructed from the initial conditions as explained in section 3, the orbit in configuration space can be obtained and all its geometrical properties established.

4.1 The case of an attractive interaction with the v-origin inside the hodograph

Let us assume that the origin of coordinates in velocity space is within the circle of the hodograph; this is the case whose realization from initial conditions was previously discussed in section 3 and was illustrated in figure 1. Please refer to figure 2 for the schematic representation of the geometric steps that follow and as an aside note that every single step can be accomplished using only straight edge and compass.

The points Q, R, O, O', P and C in figure 2 have exactly the same meaning as in figure 1, that is, they serve to construct the Hamilton vector $\overline{OC}$ and the hodograph centered at C given the initial conditions $\mathbf{r}_0$ (the straight line $\overline{QR}$) and $\mathbf{v}_0$ (the straight line $\overline{OP}$), and the vector $-\hat{\mathbf{e}}_\theta R_h$ (the straight line $\overline{OO'}$). In fact, we will always assume this meaning for the labeling of points in figures 2–5, in figure 6 the naming of points is similar excepting that O' is not found to be necessary.

To locate any point in the orbit, extend the straight line $\overline{PO}$ until it again intercepts the hodograph at point T (see figure 2a). Trace a perpendicular to $\overline{CT}$ passing through the point R, this line intercepts the symmetry axis (constructed as in section 3) at the very important point Q'. To locate the point on the orbit corresponding to any given point on the hodograph, let us notice that we have already one such pair of points, the initial conditions: the points R and P. Let us

choose another point P' on the hodograph; to begin, draw the straight line $\overline{OP'}$, extend it until it intersects the hodograph at point T'. Draw two straight lines perpendicular to $\overline{CP'}$ and to $\overline{CT'}$ passing through Q and Q' respectively; we assert that this two perpendiculars meet at the required point R' on the orbit, as was the case with the perpendiculars to the straight segments $\overline{CP}$ and $\overline{CT}$, related to the initial conditions and meeting at R. To draw the complete orbit, we have to repeat the same procedure starting from each and every point on the hodograph, in this way drawing, point by point, the whole orbit—which is shown as the gray curve which includes the points R and R' in figure 2a.

What are the properties of the just constructed orbit? The easiest way to answer this question is by establishing the orbital shape. To do this, let us first draw the circular arc $Q'W$ centered at R with a radius equal to the lenght of $\overline{RQ'}$. This arc intercepts the straight line $\overline{QO}$ at the point W (see figure 2b). Next, trace the circular arc WW' centered at Q with radius $\overline{QW}$. It is now easy to see, just by noticing that the shaded triangles $\triangle P'T'C$ and $\triangle W'Q'R'$ are both isosceles and similar to each other (this happens by construction), that the point R' on the orbit is at the same distance from the arc WW' than from the point Q'. We can see thus that the radius of the circular arc WW' is the sum of the lenghts of $\overline{QR'}$ and $\overline{Q'R'}$ and, therefore, that *in the case $E < 0$ the orbit is necessarily an ellipse* with major axis $2a$ equal to the lenght of the line segment $\overline{QW}$. The auxiliary point Q' is thus one of the foci of the ellipse, the other one coinciding with the centre of force Q. The line $\overline{QS}$ can be seen to be the symmetry axis of the ellipse as we have anticipated. In fact, the eccentricity of the ellipse is easily calculated as $\epsilon = h/R_h = OC/CP$ [2]. Thus, the famous Laplace-Runge-Lenz vector can be drawn as a straight line segment of lenght $\alpha\epsilon$ parallel to $\overline{SQ}$ —as the segment labeled $\mathcal{A}$ in figure 2a illustrates. It is to be noted that the circle $WW'W$" can be identified with the circle used by Maxwell and by Feynman in their respective discussions of the Kepler-Coulomb problem [6,8].

4.2 The case of an attractive interaction with the v-origin on the hodograph

Let us now assume that the origin of coordinates in velocity space happens to be precisely on the circle of the hodograph, as shown in figure 3. The symmetry axis $\overline{QS}$, as described in section 3, is the line perpendicular to $\overline{OC}$ which passes through the point Q. For the construction, we also need the auxiliary line $\overline{SW}$, paralell to $\overline{OC}$ and whose distance from the initial point R ($\overline{RW}$) is equal to the lenght of the segment $\overline{QR}$.

To construct the orbit, we first, just for the sake of convenience, translate the centre of the hodograph to the point Q. That is, the hodograph's centre is relocated to coincide with the centre of force. See figure 3. All references to points on the hodograph from now on, assume this new location for it. Let us choose an arbitrary point P' on the hodograph and draw the straight line segments $\overline{OP'}$ (the velocity)

and $\overline{QP'}$ (i.e. the vector $\hat{\mathbf{e}}_\theta R_h$). Draw a perpendicular to $\overline{OP'}$ passing through Q and intercepting the auxiliary line $\overline{SW}$ at the point W'. Erect $\overline{W'R'}$ perpendicular to $\overline{SW}$ and trace $\overline{QR'}$ perpendicular to $\overline{QP'}$ passing through Q. This line intercepts $\overline{W'R'}$ at R', a point on the orbit. We assert that any point constructed in this way belongs to a parabola which thus corresponds to the shape of the orbit in the case in which the v-origin sits on the hodograph, that is, in the $E = 0$ case. In fact, the assertion can be checked just by noting that the initial conditions are related in exactly the same way as we did in the previous section.

The proof that the orbit is a parabola is similar to that given for the elliptic case of subsection 4.1, based as it is on the similarity of the shaded isosceles triangles $\triangle QP'O$ and $\triangle R'QW'$ in figure 3 (please remember that we are always referring to points in the displaced (continuous) hodograph). This similarity is enough to show that the lenghts of $\overline{QR'}$ and of $\overline{R'W'}$ are the same, thus establising the orbit as the locus of points equidistant from both the point Q and the straight line $\overline{SW}$. Therefore, Q is seen to be the focus and the segment $\overline{SW}$ the directrix of the parabola. Notice that the directrix is defined by the direction of the Hamilton vector $\mathbf{h}$, being thus also parallel to the velocity at pericentre $\overline{CX}$ ($\mathbf{v}_p$). Notice also that both the hodograph and the orbit exhibit that the speed at pericentre (the lenght of $\overline{CX}$, v_p) is always greater than any other speed in the problem.

4.3 The case of an attractive interaction with the v-origin outside the hodograph

Let us now assume that the origin of coordinates in velocity space O is outside the circle of the hodograph as shown in figure 4. The Hamilton vector is $\overline{OC}$ and the hodograph is the circle centered at C with radius $\overline{CP}$. A very important difference with the cases of subsections 4.1 and 4.2, is that here the hodograph is not the whole circle since, in this case, this is the only way of guaranteeing that every point in the orbit came from just one and only one point, i.e. every point corresponds to just one velocity. In this way we also guarantee that every other speed v is always less than the speed at pericentre v_p [3–4]. This implies that, in the attractive case considered here, the hodograph is the arc of the circle 'farthest' from the origin —shown as a continuous line in figure 4a. As in the previous two subsections, the symmetry axis $\overline{QS}$, is the line perpendicular to $\overline{OC}$ and passing through the centre of force Q as illustrated in figure 4a.

For constructing the orbit, we first need to locate the auxiliary point Q' (figure 4a). To locate Q' first trace the straight line $\overline{CT}$, were T is the unphysical intercept of the hodograph with the line $\overline{OP}$, then erect on R a perpendicular to $\overline{CT}$. The intercept of this last line with the symmetry axis $\overline{QS}$ is precisely the auxiliary point Q'. With these geometric data we can begin the construction of the orbit.

Let us select any point P' on the hodograph, trace the straight lines $\overline{CP'}$ and $\overline{CT'}$. Erect perpendiculars to them passing through Q and Q', respectively, the intersection of these lines is another point R' on the orbit. It is now obvious that

for constructing the whole orbit you have to repeat this procedure over and over again, starting from each point on the hodograph, you can check that the initial conditions are related by this same procedure. The asymptotic velocities and the speed at infinity are also easy to obtain. To this end just trace, starting from the v-origin O, the straight line segments, $\overline{OB}$ and $\overline{OB'}$, tangent to the hodograph. These segments correspond, respectively, to the asymptotic velocities $\mathbf{v}_{-\infty}$ and $\mathbf{v}_{\infty}$ (as follows from angular momentum conservation), therefore, as can be seen in figure 4b, their common lenght is the sought after speed $v_\infty = \sqrt{h^2 - R_h^2}$.

We are just left with the task of establishing the shape of the orbit. To this end, trace the circular arc QW centered at R with radius QR, this arc intercepts the segment $\overline{Q'R}$ at the point W. See figure 4b. Next, trace a circle centered at the auxiliary point Q' and radius $Q'W$. Given the similarity of the two shaded isosceles triangles $\triangle CP'T'$ and $\triangle R'QW'$, we can assert that any point R' is at the same distance from Q and from the circle WW' (shown as a continuous dark circle in figure 4b), where W' is the intercept of this last circle with $\overline{Q'R'}$. From the fact that any point on the orbit is at the same distance from the point Q and from the circle WW', we can establish that the difference between the distances from R' to Q and from R' to Q', is equal to the radius of the circle WW' and, therefore, it is a constant. But this is precisely the definition of an hyperbola, which is thus the shape of the orbit in the $E > 0$ attractive case. This is illustrated in figure 4b.

4.4 The case of a repulsive interaction

In the previous sections we have been addressing the construction of orbits in the case of an attractive interaction in equation (1), i.e. the case with $\alpha > 0$; however, the sign of α does not really matter for the shape of the hodograph, it is *always* a circle. But, as we already know [13], there are nevertheless differences in the kind of motions in configuration space that are allowed. How can we understand such differences starting from just the hodograph? Finding a sort of geometric answer to this question is one of the purposes of this section.

Notice that, in the case at hand and as shown in figure 5a, both points O and P are on the same side of the straight line segment $\overline{QR}$, therefore the length of $\overline{OL}$ (the Hamilton vector $\mathbf{h}$) is greater than $\overline{OP}$ (the initial velocity $\mathbf{v}_0$) and that $\overline{OO'}$ (the hodograph radius is less than $\overline{OP}$, this means that the origin of coordinates in velocity space is always outside the circle of the hodograph. That is, whenever $\alpha < 0$ and since $\mathbf{v}_0 \cdot \hat{\mathbf{e}}_\theta$ could not be negative nor vanish, the only possibility for the v-space origin is to be outside the hodograph. In the 'modern' language of classical mechanics, if $\alpha < 0$ then the only possible motions have a necessarily positive energy.

The points Q, R, O, O', P, T and C in figure 5 have exactly the same meaning as in the previous figures 1 to 4, that is, they serve to construct the Hamilton vector $\overline{OC}$ and the hodograph centered at C, given the initial conditions $\mathbf{r}_0$, the straight line $\overline{QR}$, and $\mathbf{v}_0$, the straight line $\overline{OP}$, and the vector $-\hat{\mathbf{e}}_\theta R_h$, represented by the

straight line $\overline{OO'}$, This case is similar to that of section 4.3 since the hodograph is not the whole circle; this can be argued using essentially the same argument as in that section [2–4]. In the repulsive case considered here the hodograph is the circular arc 'closer' to the origin —which is shown as a continuous line in figure 5. As in the previous subsections, the symmetry axis $\overline{QS}$ is the line perpendicular to $\overline{OC}$ and passing through the centre of force Q.

To find the orbital shape we need the auxiliary point Q', which is the intercept of a perpendicular to $\overline{CT}$ going through R with the symmetry axis $\overline{QS}$. Now is just a matter of choosing an arbitrary point P' on the hodograph, and prolonguing the straight line segment $\overline{OP'}$ until it again meet the hodograph at point T'. Trace the straight line segments $\overline{CP'}$ and $\overline{CT'}$ and erect perpendicular segments going through Q and Q', respectively. The intercept of these perpendiculars is the corresponding point R' on the orbit. Repeating the procedure for every point on the hodograph we can obtain the whole orbit. The orbit is again, as in section 4.4, an hyperbola with foci Q and Q', as can be shown by considering that any point on the orbit is at the same distance from the fixed point Q' and from the auxiliary circle WW' centered at Q, defined as in section 4.4. The complete argument uses the similar isosceles triangles $\triangle CP'T'$ and $\triangle R'W'Q'$ and essentially repeats the argument of the previous section.

4.5 The Rutherford problem

Let us pick the point Q as the location of the repulsive centre of force. To describe geometrically a scattering situation, we have basically the same situation of sections 4.3 and 4.4, the only difference being that, here we are given the velocity $\mathbf{v}_{-\infty}$, i.e. the velocity evaluated at a time in 'the infinitely distant past' and the impact parameter b, not the velocity and the position at a certain finite time t. See figure 6. With the data just mentioned and from the location of the centre of force Q, draw the line segment $\overline{OK}$ parallel to $\mathbf{v}_{-\infty}$, starting from the arbitrary point O but passing at a distance b off the centre of force.

If on $\overline{OK}$ we choose the segment $\overline{OB}$ to represent $\mathbf{v}_{-\infty}$, the point O would have been implicitly selected to play the role of the v-origin. Then, from the point B, erect a perpendicular straight line segment, of lenght R_h, up to the point C. Next, centered at C draw a circle with radius $\overline{CB}$, a part of this circle is the hodograph of the problem. If we draw the tangent to the circle $\overline{OB'}$, this represents the asymptotic outgoing velocity at infinity $\mathbf{v}_{\infty}$; the hodograph is thus the circular arc BPB' and the Hamilton vector is the line segment $\overline{OC}$ bisecting the angle $\angle B'OB$. This angle is usually called the deflection angle ξ. In fact, the right triangle $\triangle OBC$ gives immediately the Rutherford relation between $\xi/2$ and L

$$\tan\frac{\xi}{2} = \frac{R_h}{\sqrt{h^2 - R_h^2}} = \frac{\alpha}{v_{-\infty}L}, \tag{6}$$

which can be used as the starting point to derive the famous Rutherford scattering formula [3,8,13]. See also [4] where the Rutherford problem is discussed taking a velocity space point of view from the start.

5. Conclusions

We have exhibited that the orbits of the Kepler-Coulomb problem can be obtained and classified (basically in terms of the energy) starting from the hodograph and using techniques of synthetic geometry requiring no more than straight edge and compass. We have exhibitted that the Hamilton vector is crucial for deciding geometrically if the orbits are bounded or not and, furthermore, that with its help, we can draw point by point any orbit whatsoever. On the other hand, speaking on the purely geometrical content of the paper, we have managed to offer an admittedly not very systematic proof of an elementary but not widely known geometric result, namely, that the conic sections can be defined as the locus of points equidistant from both a fixed point and a fixed circle. The geometric method can be further justified as in [15].

We have learnt a lot in trying to do mechanics using the nowadays non-standard methods of Newton. We hope that this article may convey to the readers the aesthetic pleasures we discovered in the geometric structure of Newton's mechanics. We think these considerations are enough to justify the approach presented in this article which exhibit the enormous power of geometric reasoning in classical mechanics [16]. However, we have to emphasize that Newton's geometric methods go far beyond the simple results obtained here; it has been discovered, for example, that the Principia contains, among other things, astonishing geometric proofs of deep results on the properties of Abelian integrals [17].

Acknowledgements

This work has been partially supported by CONACyT (grant 1343P-E9607). ALSB and HNNY want to acknowledge very interesting discussions with Leticia Fuchs Gómez. We must also thank our colleagues D. Moreno, J. A. González Menéndez, and R. W. Gómez González for their very useful comments. This paper is dedicated with thanks to Q. Kuro, Q. Sieri, M. Tlahui, M. Miztli, G. Tigga, B. Caro, Ch. Ujaya, U. Sasi and F. C. Minina. AGV wants to express his thanks to Luis González y González and Armida de la Vara for all the support and encouragement they have always given him. We should also thank a careful referee that pointed out several misprints and omissions in the original version of the article and that shared with us our love for geometric argumewnts.

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Figure Captions

Figure 1
The geometrical procedure for obtaining the Hamilton vector and the hodograph from given initial conditions $\mathbf{r}_0$ and $\mathbf{v}_0$ is illustrated. O labels the origin of coordinates in velocity space or v-origin and Q labels the location of the centre of force. To draw the segment OO', corresponding to $-\hat{\mathbf{e}}_\theta \alpha/L$, we assumed that $\mathbf{L}$ points outside the plane of the paper. The Hamilton vector is the line segment $\overline{OC}$, the circle $X'PX$ centered at C is the hodograph. The straight line segments $\overline{SX}$ and $\overline{QS}$ correspond, respectively, to the dynamical symmetry axis of the orbit and of the hodograph. The discussion related to this figure can be found in section 3.

Figure 2
2a. The procedure for reconstructing the orbit when the hodograph encompass the v-origin is illustrated. The meaning of the points marked Q, R, O, O', P and C is illustrated. Notice that the orbit is indeed closed; furthermore, notice that despite appearances the point S does **not** necessarily correspond to the vertex of the ellipse. For the detailed discussion of this case see the section 4.1.

2b. To demonstrate the orbit is indeed an ellipse we need to recognize that the two shaded isosceles triangles $\triangle Q'R'W'$ and $\triangle CP'T'$ are similar to each other.

Figure 3
The procedure for reconstructing the orbit when the v-origin is precisely on the hodograph is illustrated. For the sake of convenience, let us first translate the whole hodograph from its original place centered at C to a new location centered at Q (the centre of force). *All references to points on the hodograph are to be understood at its displaced location.* To demonstrate that the orbit is a parabola we only need to recognize that the two shaded isosceles triangles $\triangle QP'O$ and $\triangle R'QW'$ are similar to each other. Notice that the straight line segment $SW'W$ corresponds to the auxiliary circle of the previous figure. Thus, from this point of view, the directrix is just a degenerate circle with infinite radius. See the discussion in section 4.2.

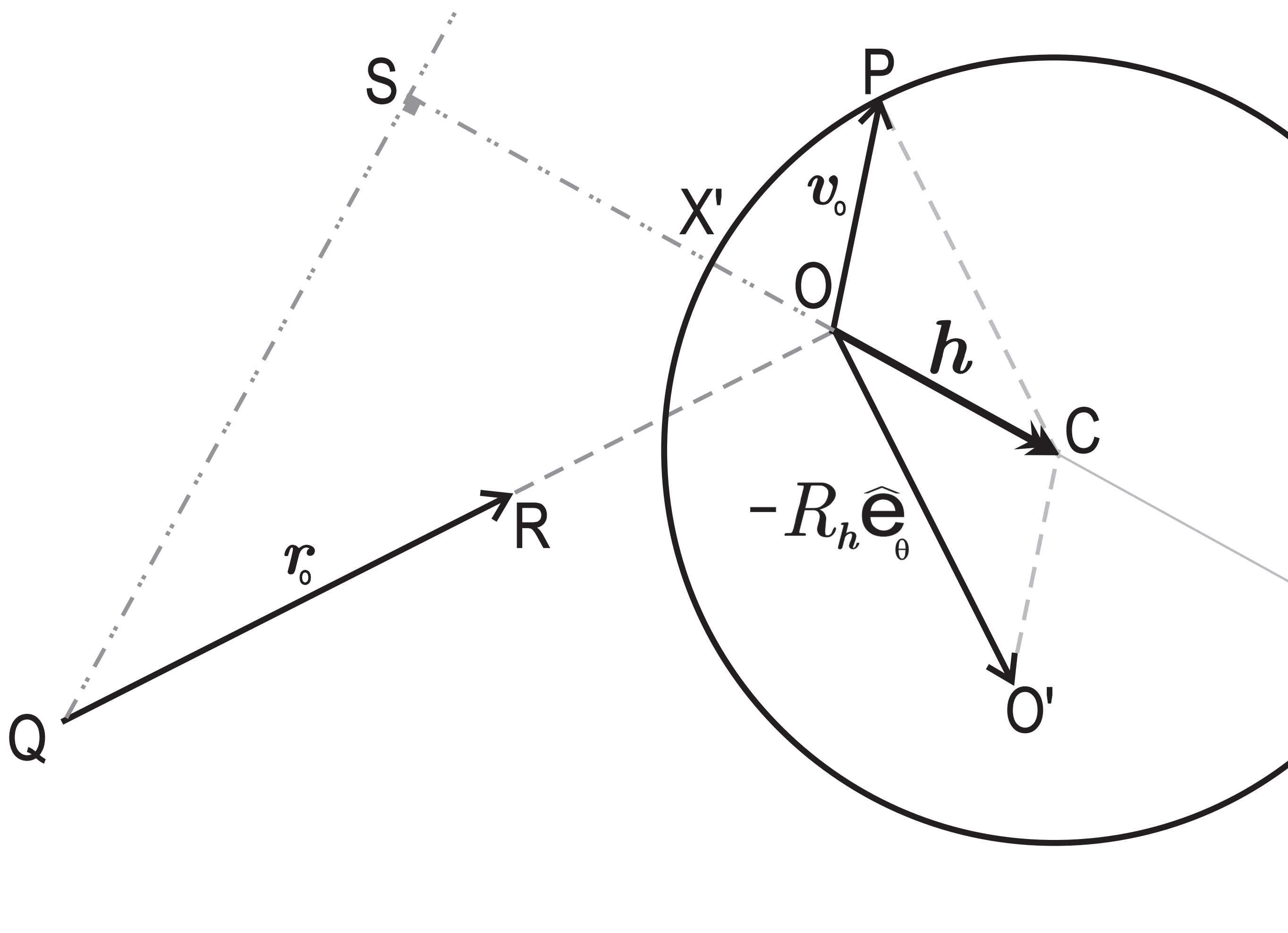
Figure 4
4a. The procedure for reconstructing the orbit when the v-origin is outside the hodograph is illustrated. The points P and P' on the hodograph correspond to the points R and R' on the orbit.See section 4.3

4b. To demonstrate that the orbit is an hyperbola whose internal focus coincides with the centre of force, we only need to recognize that the two shaded isosceles triangles $\triangle CP'T'$ and $\triangle R'QW'$ are similar to each other.

Figure 5
The procedure for reconstructing the orbit starting with finite initial conditions. Here we consider the case when the v-origin is outside the hodograph and the in-

teraction is repulsive. The two shaded triangles are important for the discussion in section 4.4.

Figure 6
The Rutherford relation between ξ and L can be simply obtained from the hodograph as we illustrate in this figure. We also exhibit the procedure for reconstructing the orbit in a *scattering* situation. We consider the case when both the v-origin is outside the hodograph (i.e. the case $E > 0$) and the interaction is repulsive (i.e. $\alpha < 0$). For a brief discussion see section 4.5. A complete discussion from the point of view of velocity space can be found in Ref. 4.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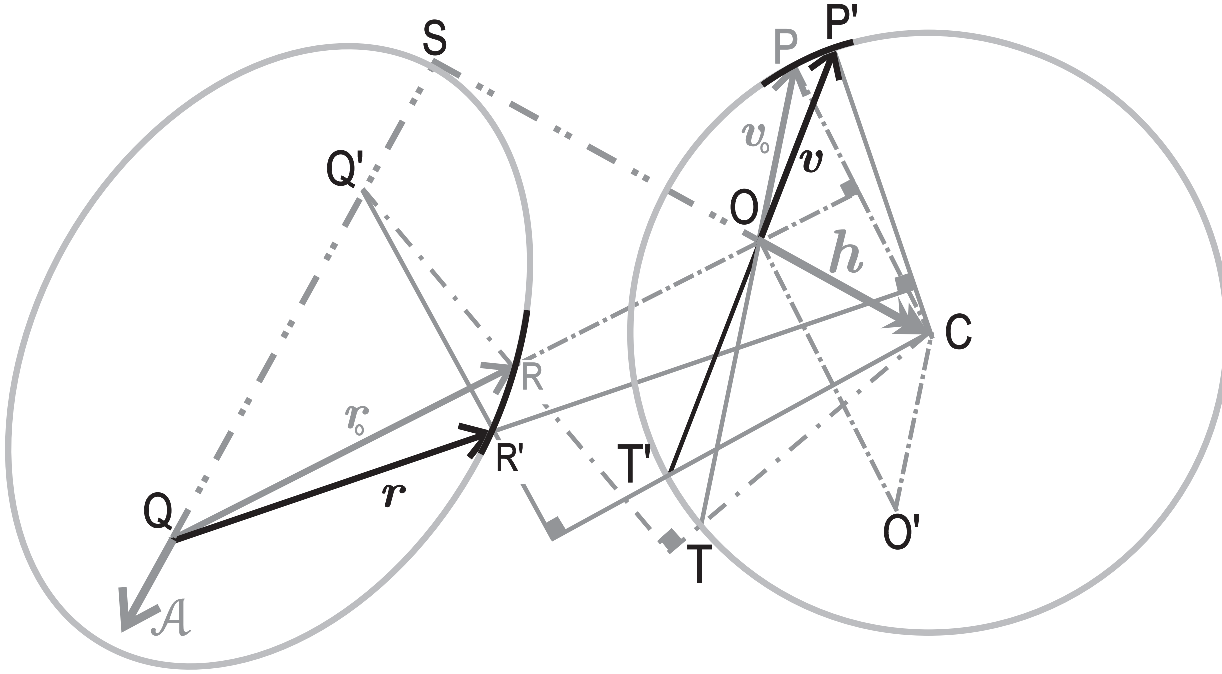

fig 2a Geometry of kepler problem. Elliptic motion

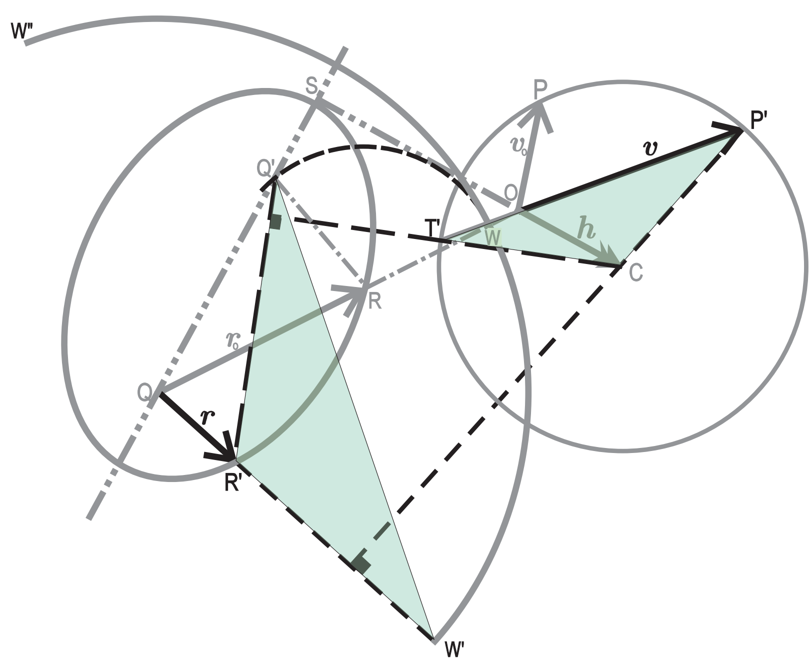

fig 2b geometry of kepler problem. Elliptic motion

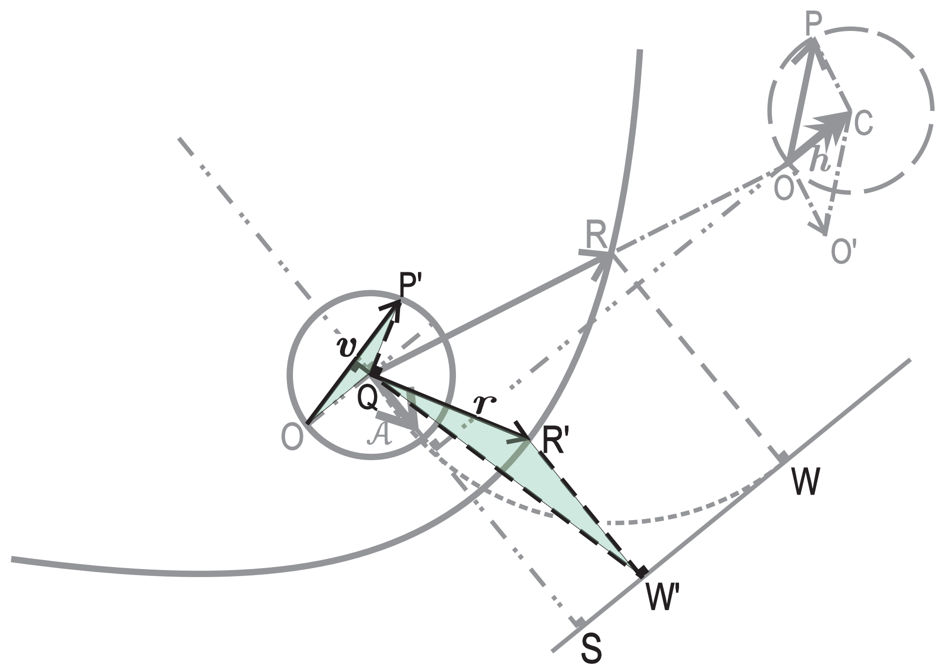

fig 3 geometry of kepler problem: parabolic motion

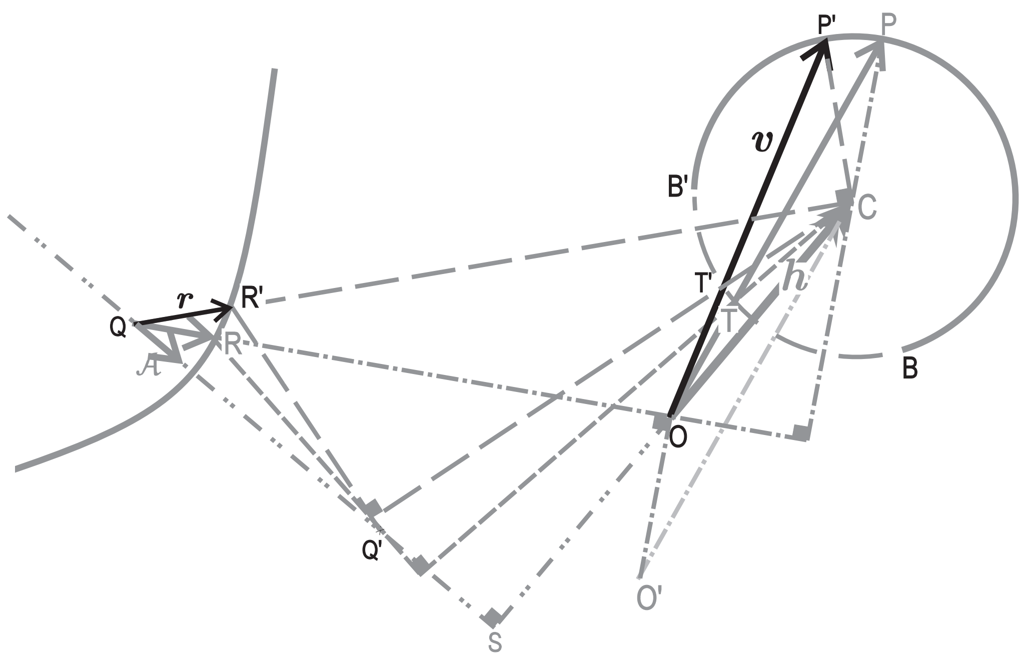

fig 4a Geometry of kepler problem. Hyperbolic motion (attractive)

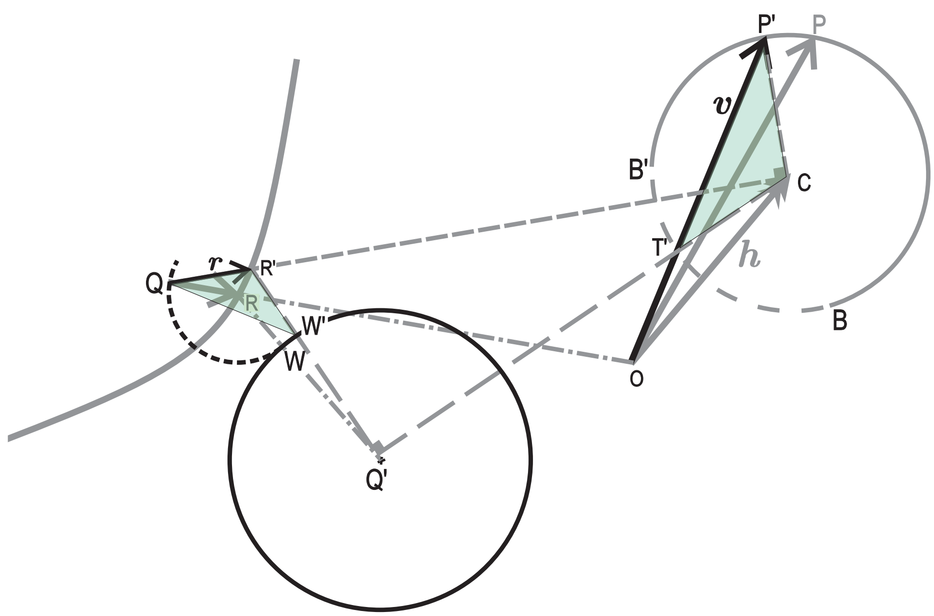

fig 4b geometry of kepler problem: hyperbolic motion (attractive)

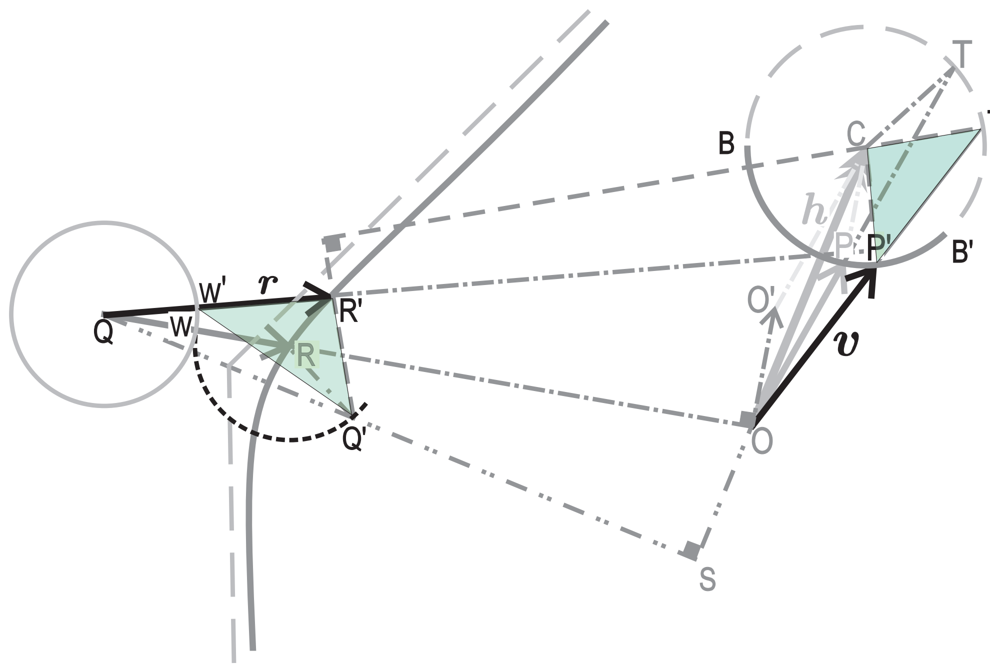

fig 5 geometry of kepler problem: hyperbolic motion (repulsive)

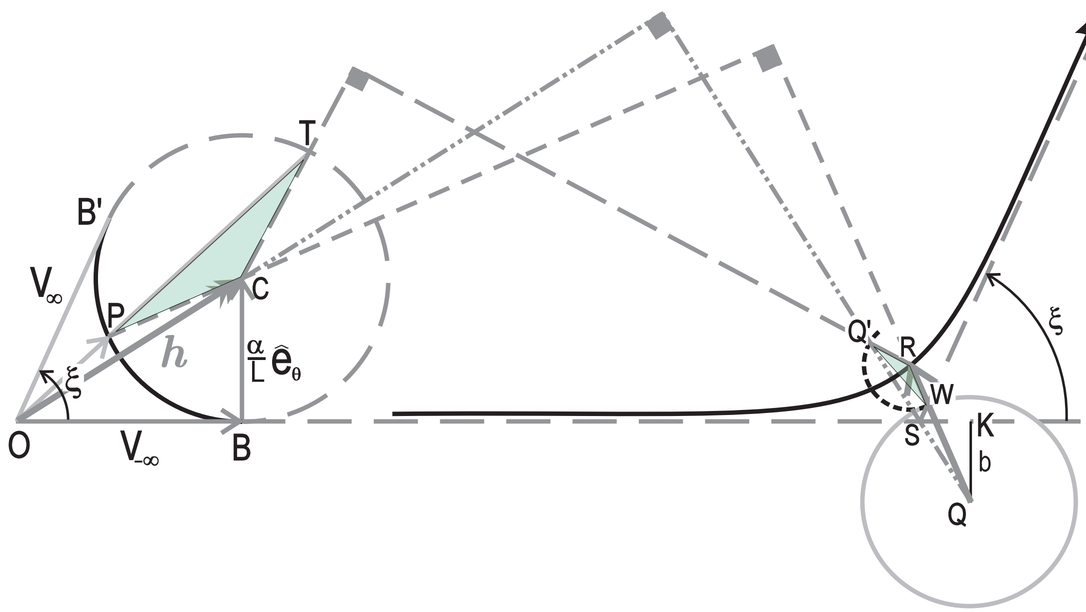

fig 6 geometry of kepler problem: hyperbolic motion (repulsive)
Rutherford scattering